RESONANT GENERATION OF HARMONIC THERMAL WAVES IN MEDIA WITH MEMORY

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A problem which is, in principle, new, on the possibility of resonant generation of harmonic thermal waves in media with thermal memory, is posed and solved. The conditions for stable generation of such waves are elucidated.

Introduction. Recently, there has been great interest in studies of thermal transport processes in media with thermal memory (hereditary thermal media). On the one hand, this is related to the great number of such processes in various fields: heat transport in pure crystals and dielectrics at low temperatures (ballistic propagation and second sound); thermal propagation during the action of laser irradiation on a metal; heat transport in regions of transitions to a glassy state for organic liquids and polymers; and heat transport in disperse media. On the other hand, this interest is related to predicted new effects in the processes of hereditary heat transport, and also to the possibility of formulating basically new problems, which are inherent in these media alone. Thus wave effects of the propagation of heat in hereditary media and the possibility, in principle, of the existence of weakly damping and amplifying thermal media with memory [1] results in the possibility of developing wave thermophysics (analogs of many optical and radiophysics wave phenomena and devices). Below it will be shown that in thermal media with memory, it is possible to generate resonant harmonic thermal waves (TW) of fixed frequency (modes).

The frequency of the excited TW depends on the thermophysical properties of the memory of the medium, the geometry and conditions of heat exchange of the body, and also the intensity of the applied external fields as well, which create the amplifying thermal medium. This phenomenon is analogous to the resonant generation of electromagnetic waves in an optical quantum generator (OQG), which analogy also applies to the elements of the OQG necessary for wave generation.

Working Principle of a Thermal Resonant Generator. In order to work, an OQG must have an active (inverted population) optical medium [2], and also an open Fabry–Perot optical resonator which provides positive feedback and allows selection of certain modes from the generator contour of excited emission lines. In a thermal resonant generator (TRG), the active working medium is a specially selected hereditary thermal medium with volume sources distributed in it which are proportional to the temperature. In [1, 3] it was shown that for the transit of TW through such a medium, the TW amplification coefficient $\exp(-\xi(\omega)x)$ can be greater than one (which corresponds to $\xi(\omega) < 0$). The region of TW amplification $\omega\in(0, \omega_{cr})$ is the thermal analog of the contour of optical excitation lines. Their construction will be discussed in detail below.

The thermal analog of open optical resonance is a rod with thermally insulated lateral surface and end face $x = 0$. At the other end of the rod $x = \ell$, a boundary condition of the third type is prescribed, which simulates the outflux of energy to the external medium. The rod is made of an active amplifying thermal-hereditary medium, as described in [1, 3]. In practice, to realize this medium, a sufficiently powerful external field is created (electrical, electromagnetic, etc.). With the help of this field, energy pumping occurs at every point in the rod in accordance with the chosen pumping mechanism. Under certain conditions described below, such a thermal system has a positive thermal feedback and is the thermal equivalent of an optical resonator with a population inversion in its medium. In it, it is possible to generate modes in some TW frequency range $\omega\in(0, \omega_{cr})$. In this case, random temperature fluctuations at the rod boundary $x = \ell$ give an initial broad-band TW spectrum distribution $\omega\in(0, \infty)$, but in the process of evolution there arise modes, which are the resonant

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modes of the resonator (rod). This investigation is made in a linear approximation, and therefore nonlinear effects will not be considered, which limits the amplitude of the TW generated. Unlike OQG, resonant generation in TRG can be accompanied by asymptotic growth (for $t \to \infty$) of the temperature field: thermal explosion. Therefore in analyzing TRG operation, it is necessary to consider the stability condition for resonant TW generation.

Mathematical Model for TRG Operation. The mathematical model for the physical problem which has been posed is formulated as follows. The governing relation for the hereditary thermal medium and the conservation equation for the internal energy have the form

$$q(x,\ t) = -\lambda_0\left[\lambda_1(0)\frac{\partial u}{\partial x} + \int\limits_0^\infty \dot{\lambda}_1(\tau)\,\frac{\partial u(x,\ t-\tau)}{\partial x}\,d\tau\right];$$

$$e(x,\ t) - e_0 = \frac{\lambda_0}{a_0}\left[c_1(0)\left(u(x,\ t) - u_i\right) + \int\limits_0^\infty \dot{c}_1(\tau)\left(u(x,\ t-\tau) - u_i\right)d\tau\right]; \qquad (1)$$

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + \sigma(x,\ t);\ \ \sigma = \sigma_1 + \sigma_2\left(u(x,\ t) - u_i\right).$$

The conditions of uniqueness consist of the initial condition $u(x, 0) = 0$ and $\partial u(x, 0)/\partial t = 0$ for Maxwell media, the condition of thermal insulation $q(0, t) = 0$ at the boundary $x = 0$, and the boundary condition at $x = \ell$, which simulates the energy outflux:

$$q(l,\ t) = \alpha\left[u(l,\ t) - u_0(t)\right];\ \ u_0(t) = \exp(i\omega t). \qquad (2)$$

The model of TRG operation with $\alpha > 0$ is analogous to OQG operation in the continuous generation mode. It is assumed that at $t = 0$, a δ-shaped temperature fluctuation with an amplitude of one takes place at the $x = \ell$ boundary. By considering an expansion of $\delta(t)$ in the form of a Fourier integral $\delta(t) = 1/2\pi \int_{-\infty}^{\infty} \exp(i\omega t)d\omega$ and the linearity of the problem, it is possible to examine separately the evolution of each mode $\exp(i\omega t)$, $\omega \in (0, \infty)$. This corresponds to giving $u_0(t)$ in the form of (2).

General Expression for the Solution of the Problem. Applying the Laplace transform to (1), (taking into account the conditions of uniqueness), the solution to the problem in the image domain (denoted by the corresponding capitalized letters) can be written in the form

$$U(x,\ p) = \left[\frac{u_i}{p} + \frac{r_1}{(p\,(p^2C_1(p) - r_0)}\right]\left[1 - \frac{1}{D_0(p)}\operatorname{ch} h(p)\,x\right] + \frac{U_0(p)}{D_0(p)}\operatorname{ch} h(p)\,x;$$

$$D_0(p) = \operatorname{ch} h(p)\,l + \frac{R_T}{Z(p)}\operatorname{sh} h(p)\,l;\ \ h(p) = \sqrt{\frac{K(p)}{a_0}};\ \ K(p) = \frac{pC_1(p)}{\Lambda_1(p)}; \qquad (3)$$

$$Z(p) = \sqrt{p^3\left[C_1(p) - \frac{r_0}{p^2}\right]\Lambda_1(p)/\lambda_0\rho_0 c_0};\ \ r_0 = \frac{a_0}{\lambda_0}\sigma_2;\ \ r_1 = \frac{a_0}{\lambda_0}\sigma_1.$$

In (3), we pass over to the original form, using the theory of residues. The solution to the posed problem can be represented in the form of a sum:

$$u(x,\ t) = u_s(x) + u_2(x,\ t) + u_3(x,\ t); \qquad (4)$$

$$u_s = \sum_{p_l}\exp(p_l t)\left[1 - \frac{1}{D_0(p_l)}\operatorname{ch} h(p_l)\,x\right]\operatorname*{Res}_{p_l}\left\{\frac{u_i}{p} + \frac{r_1}{p\,[p^2C_1(p) - r_0]}\right\}; \qquad (5)$$

$$u_2(x,\ t) = \sum_{p_n}\exp(p_n t)\left[U_0(p_n) - \frac{u_i}{p_n} - \frac{r_1}{p_n\,(p_n^2C_1(p_n) - r_0)}\right]\operatorname*{Res}_{p_n}\left\{\frac{1}{D_0(p)}\operatorname{ch} h(p)x\right\}; \qquad (6)$$

$$u_3(x,\ t) = \sum_{p_m}\exp(p_m t)\left[\frac{1}{D_0(p_m)}\operatorname{ch} h(p_m)\,x\right]\operatorname*{Res}_{p_m}\{U_0(p)\}. \qquad (7)$$

In (5)-(7), the symbol Res{·} means that only the singular points (poles) of the expression in the curly brackets enter into the equation. The first term describes the steady temperature field, which arises in the slab due to the initial temperature, and also due to the continually acting sources.

Using the properties of the relaxation function for $p \to 0$, $\Lambda_1(p) = C_1(p) = p^{-1}$ [1, 3], the steady part of the solution can be written in the form

$$u_S(x) = \left(u_i - \frac{r_1}{r_0}\right)\left[1 - \frac{1}{D_0(0)} \cos\sqrt{\frac{r_0}{a_0}}\, x\right];$$
$$D_0(0) = \cos\sqrt{\frac{r_0}{a_0}}\, l - \sqrt{\frac{r_0}{a_0}}\, \frac{l}{\mathrm{Bi}} \sin\sqrt{\frac{r_0}{a_0}}\, l. \tag{8}$$

The memory of the medium does not influence $u_S(x)$. For $r_0 \to 0$ ($\sigma_2 \to 0$), (8) transforms to a well-known steady temperature distribution [4].

The second term $u_2(x, t)$ corresponds to the stability (for $t \to \infty$) of TRG operation, and therefore its analysis gives us the condition of asymptotic stability (without thermal explosion) of the working regime of the TRG. The third term $u_3(x, t)$ describes the unsteady temperature field due to the effect of the external action $U_0(p)$. Its analysis gives the condition of interest for resonant thermal generation. Its compatibility with the conditions of TRG stability determines the possibility of realizing TRG in principle.

Analysis of the Condition for Resonant Thermal Generation. The term $u_3(x, t)$ is obtained for the inversion of (3) to the original space. For this, only the poles $\mathrm{Res}_{p_m}\{U_0(p)\}$ due to the external action of the temperature at the boundary $x = \ell$ are considered. This part of (3) is conveniently represented in the form of an expansion in terms of exponentials

$$U_3(x, p) = \frac{\alpha Z(p) U_0(p)}{1 + \alpha Z(p)} \sum_{n=0}^{\infty} R_0 \exp(-2nlh(p)) [\exp(-(l-x) h(p))$$
$$+ \exp(-(l+x) h(p))]; \quad R_0(p) = \frac{1 - \alpha Z(p)}{1 + \alpha Z(p)}. \tag{9}$$

Expansion (9) corresponds to a representation of the transmitted and reflected thermal waves in the transformation, with every term in (9) containing two waves: 1) $\exp[-(l - x)h(p)]$ propagating from the boundary $x = \ell$ to $x = 0$; and 2) $\exp[-xh(p)]$, a wave propagating from $x = 0$ to $x = \ell$. In (9), $R_0(p)$ is the sum of the reflection coefficients from the first and second boundaries for the image temperature. Since at each boundary, (5) for the transformed temperature and the thermal flux reflection coefficient are equal in magnitude and opposite in sign, $R_0(p)$ is the same for the temperature and the flux. The factor $\exp(-\ell h(p))$ is the extinction coefficient for a single transit of the thermal wave through the slab (which is also identical for the temperature and the flux). Assuming that there exists only one pole in this problem in the original domain, the function $u_3(x, t)$ for $x = \ell$ takes the form

$$u_3(l, t) = \frac{\alpha Z(i\omega) \exp(i\omega t)}{1 + \alpha Z(i\omega)} \sum_{n=0}^{n_0} [|R_0(i\omega)| \exp(-2l\xi(\omega))]^n \exp(in\Phi_0(\omega))$$
$$\times \left\{1 + \exp\left[-2l\xi(\omega) - \frac{i2l\omega}{w(\omega)}\right]\right\}; \quad \Phi_0(\omega) = \Phi(\omega) - \frac{2l\omega}{w(\omega)}; \tag{10}$$
$$\xi = \frac{\pm 1}{\sqrt{2a_0}} [\mathrm{Re}\, K(i\omega) + |K(i\omega)|]^{\frac{1}{2}}; \quad w = \frac{\sqrt{2a_0}\,\omega}{[|K(i\omega)| - \mathrm{Re}\, K(i\omega)]^{\frac{1}{2}}};$$
$$\Phi = \arg R_0(i\omega).$$

In (10), $n_0 = \infty$ for media of the Fourier type, while for Maxwell media n_0 is determined as in [3]; $\Phi_0(\omega)$ is the total change in the phase of the thermal wave for a single transit of the slab and reflection from its ends; $\xi(\omega)$, $w(\omega)$ are the extinction coefficient and the propagation velocity of the TW.

Expansion (10) makes it possible to formulate the phase and amplitude conditions for the amplification of longitudinal TWs, which are analogous to the conditions of amplification for longitudinal modes in a OQG in the Boyd and Gordon approach [2]. The condition for amplification in terms of amplitude is conveniently written in the form

$$\xi(\omega) < \frac{1}{2l} \ln |R_0(i\omega)|. \tag{11}$$

The condition for phase repetition

$$\Phi_0(\omega) = 2\pi m, \ m = \pm 1, \ \pm 2, \ \ldots, \tag{12}$$

depends on the dimensions of the rod, equilibrium thermophysical and relaxation properties of the material, on the external energy pumping (the coefficient r_0) and on the condition of heat exchange (the coefficient α). Since $|R_0(i\omega)| \leq 1$, condition (11) is more strictly analogous to the condition $\xi(\omega) < 0$, obtained in [3] for a semi-infinite rod, and is identical with it in the case of ideal thermal contact $\alpha \to \infty$, $|R_0(i\omega)| = 1$. In the case of ideal thermal contact, (boundary condition of the first type), the "excitation contour" is bounded by the axis $O\omega$ and the curve $\xi(\omega)$, with $\xi(\omega) < 0$, $\omega \in (0, \omega_{cr})$. In this case, the amplitude condition (11) for TW generation coincides with the condition of TW amplification, analyzed in detail in [3].

Since for ideal thermal contact $(\alpha \to \infty) R_0(i\omega) = -1$; $|R_0(i\omega)| = 1$; $\Phi = \pi$, the phase condition (12) selecting the longitudinal modes takes the form

$$\omega_m = \frac{\pi (2m - 1) w(\omega_m)}{2l}, \ m = 1, 2, \ldots \tag{13}$$

Unlike an optical medium, a thermal medium is highly dispersive (sensitive dependence of $w(\omega)$).

The simultaneous satisfaction of (11) and (12) leads to the selection and generation of one or several TW modes, excited in the TRG from the general "excitation contour" for the TWs. In this case, either a finite or infinite number of TW modes can be generated, depending on the "excitation contour." Let us analyze in more detail the case of ideal thermal contact. In an ordinary Fourier medium, TW generation is not possible. In any medium of the Fourier type, only a finite number of longitudinal TW modes can be generated, since the TW "excitation contour" is limited in frequency to $\omega \in (0, \omega_{cr})$. Both bounded and unbounded numbers of modes can be generated [3] in Maxwell media. In the standard Maxwell medium, corresponding to a hyperbolic equation, only an infinite number of modes can be generated. The amplitude of the generated waves rapidly grows in time to become infinite (but in reality is limited by nonlinear effects in TRG). As a result of satisfying conditions (11), (12), the solution $u_3(x, t)$ can be represented in the form

$$u_3(x, t) = \frac{\alpha Z(i\omega_m)}{1 + \alpha Z(i\omega_m)} \exp(i\omega_m t) \left\{ \exp\left[-(l - x)\xi - i\omega_m \frac{l - x}{w} \right] + \exp\left[-(l + x)\xi - i\omega_m \frac{l + x}{w} \right] \right\} \sum_{n=0}^{n_0} K_{TW}^n; \tag{14}$$

$$K_{TW}(\omega_m) = |R_0(i\omega_m)| \exp(-2l\xi(\omega_m)); \ \omega_m \in (0, \omega_{cr}).$$

Stability Analysis of TRG Operation. The term $u_2(x, t)$ is responsible for the stability of TRG operation. It arises as a result of the zeroes of the function $D_0(p)$, that is, p_n is the root of the equation

$$\operatorname{cth} h(p) l + \frac{1}{\alpha Z(p)} = 0, \tag{15}$$

which can be put in the form

$$K(p) = -\frac{a_0}{l^2} \mu_n^2 (\tilde{Bi}); \ \ \tilde{Bi} = Bi / p\Lambda_1(p). \tag{16}$$

Here, the correspondence function $K(p)$ has different forms for Fourier and Maxwell thermal media; μ_n is the root of the equation

$$\operatorname{ctg} \mu = \mu / \tilde{Bi}. \tag{17}$$

Equation (1) itself, source r_0, the memory of the medium, the problem geometry, and the conditions of heat exchange all influence the root p_n. Equation (15) for p_n can result in either real roots or a pair of complex-conjugate roots, which result in real expressions in $u_2(x, t)$ as well. In analyzing (15) or (16), only the case Re $p_n < 0$ for all roots p_n, corresponds to the stable damped solution ($u_2(x, t) \to 0$ for $t \to \infty$). Since every value of n in (16) may correspond to several roots (this depends on the specific form of K(p)), we will denote them by $p^{[k]}_n$, where the superscript k denotes the number of roots corresponding to the fixed value n.

In an ordinary Fourier medium, $\lambda_1(t) = c_1(t) = H(t)$ with volume source $\sigma = r_0 u$, it is not possible to generate TWs: only thermal explosion is possible. In a Fourier medium of more complex form

$$\lambda_1(t) = 1 - (1 - \lambda_1(0)) \exp(-\beta_0 t);\ p_1 = \frac{\beta_0}{\lambda_1(0)};\ \gamma_0 = \frac{1}{\lambda_1(0)};$$
$$c_1(t) = 1 - (1 - c_1(0)) \exp(-\beta_0 t);\ d_1 = c_1(0)/\lambda_1(0);$$
$$d_0 = (1 - d_1) p_1;\ h_1 = -d_0 p_1;\ \gamma_1 = p_1(1 - \gamma_0). \tag{18}$$

to everv index n in (16) there correspond two roots: $p_n^{[1]}$, and $p_n^{[2]}$. By representing the root $p_n[k]$ in the form $p_n[k] = -p_1 + \delta n[k]$ and using (16), we obtain an expression for $\delta_n[k]$

$$\delta_n^{[k]} = \frac{\beta_n}{2d_1}\left(-1 \pm \sqrt{1 - 4d_1(h_1 - \gamma_1 r_0)/\beta_n^2}\right),$$
$$\beta_n = (d_0 - r_0\gamma_0) - d_1 p_1 + a_0 \mu_n^2 / l^2. \tag{19}$$

For ideal thermal contact

$$\mu_n^2 = \pi^2 (n - 1/2)^2,\ n = 1, 2, \ldots \tag{20}$$

From an analysis of (19) and (20), it is evident that the region of stable TRG operation corresponds to the condition $\beta_n > 0$, which gives a bound on r_0:

$$\frac{r_0}{\beta_0} < 1 + \theta - 2d_1;\ \theta = \frac{1}{p_1}\frac{a_0}{l^2}\frac{\pi^2}{4}. \tag{21}$$

Here the most strict condition, n = 1, has been adopted. The region of TRG stability depends on the equilibrium thermophysical properties of the material, the thickness of the slab, and the properties of the medium memory. Decreasing the thickness of the slab, it is possible to adequately expand the region of stable TRG operation.

For the special case of medium (18) with $c_1(0) = 1$, [3] obtained in essence the amplitude condition for TW generation for ideal thermal contact. The region of its compatibility with (21) is determined by the inequalities

$$\frac{1}{1 - \lambda_1(0)} < \frac{r_0}{\beta_0} < 1 + \theta - 2d_1;\ \theta_{cr} = \frac{2 - \lambda_1(0)}{\lambda_1(0)(1 - \lambda_1(0))} - 1. \tag{22}$$

Stable TW generation is possible only in some regions of the parameter $\theta > \theta_{cr}$. Using [3], it is possible to obtain a relation between the parameters θ, $\lambda_1(0)$ and ω_{cr}: the bounds of TW generation (by the amplitude criterion):

$$\theta_{cr} < \frac{4}{\pi^2}\left[\frac{\left(\frac{\omega_{cr}}{p_1}\right)^2 + 1}{1 - \lambda_1(0)} + \frac{2}{\lambda_1(0)} - 1\right] \leqslant \theta. \tag{23}$$

In the generation region, there must be at least one natural mode of the resonator, which is determined by the phase condition (13):

$$\omega_1 = \frac{\pi}{2l} w(\omega_1);\ \omega_1 \in (0,\ \omega_{cr}). \tag{24}$$

The joint solution of (22) and (24) makes it possible to determine θ and r_0/β_0. Thus, it has been shown that in a Fourier thermal medium with memory (18), stable resonant thermal generation is possible.

It can be shown that for ideal thermal contact in the case of the standard Maxwell thermal medium (hyperbolic equation of thermal conductivity), TW generation (according to amplitude criterion (12)) and thermal explosion occur simultaneously. Physically, this is obvious since the frequency range of TW amplification in such a medium is unbounded $\omega \in (0, \infty)$, which causes the generation of infinitely many modes. Evidently, in the most general Maxwell thermal media (for more rapid relaxation of the internal energy compared to the thermal flux), stable TW generation is possible.

Conclusions. It has been shown that in the usual Fourier medium and the standard Maxwell medium, stable TW generation is not possible. In media of the Fourier type of a more general form and evidently, in certain Maxwell media, it is possible to generate stable, resonant TWs. A more precise TRG model must account for the conditions of heat exchange at the lateral surface of the rod, and the nonlinear and multidimensional nature of the problem.

Thus, it has been proved that it is possible in principle to create a resonant thermal generator. To create a TRG in practice requires a broad experimental investigation into the area of thermophysics of media with thermal memory, especially concerning the creation of amplifying thermal media.

NOTATION

u, temperature; q, thermal flux density; e, e_0, volumetric density of the internal energy and its initial value; σ, power of the internal energy sources; ℓ, slab thickness; α, coefficient of heat exchange; $\lambda_1(t)$, $c_1(t)$, relaxation functions for the heat flux and the internal energy; $\Lambda_1(p)$, $C_1(P)$, their Laplace transforms; p, Laplace variable; λ_0, a_0, equilibrium coefficients of thermal conductivity and thermal diffusivity; ω, angular frequency; u_i, initial temperature in the system; $R_T = 1/\alpha$, thermal resistivity; $Bi = \alpha\ell/\lambda_0$, Biot number; ξ, w, extinction coefficient and the propagation velocity of harmonic TWs; Φ_0, total change in the TW phase in one transit of the pane and reflection from its ends; Φ, shift of TW phases at the ends of the slab.

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